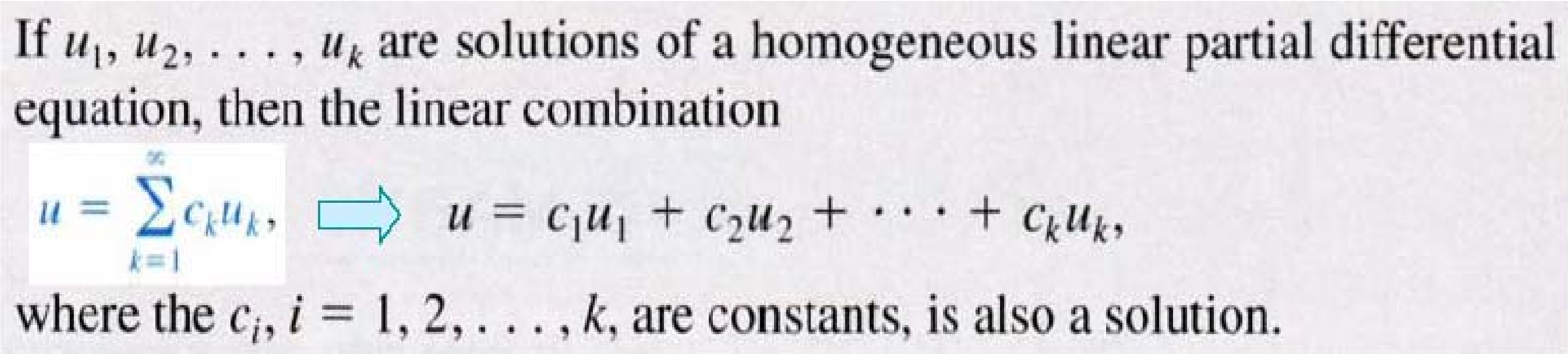
PARTIAL DIFFERENTIAL EQUATIONS: PART-2

One-dimensional Wave Equation

Lecture #14

Superposition principle

Before we take a closer look at some of the important PDE occurring in applications such as wave equations and heat equations, let us recall a theorem in linear equations.



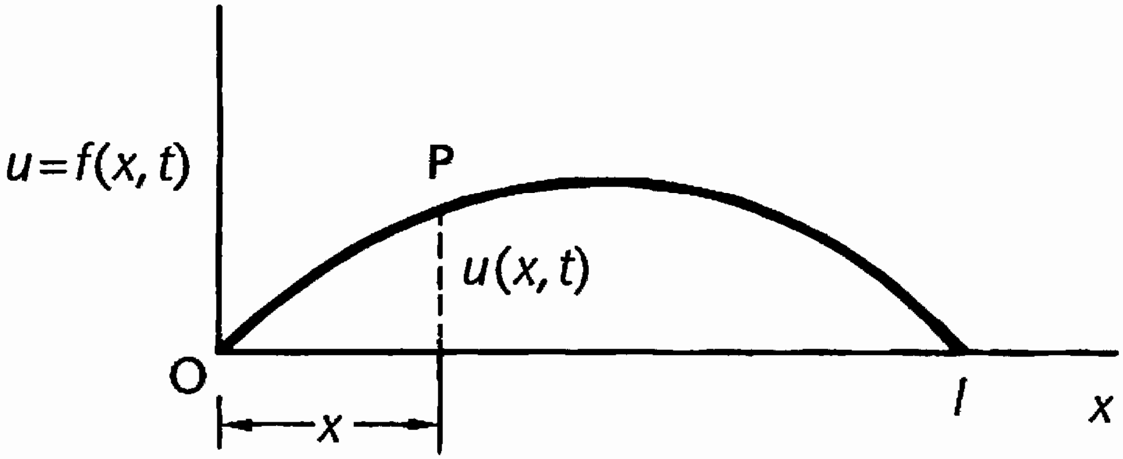
If $u_1, u_2, \ldots, u_k$ are solutions of a homogeneous linear partial differential equation, then the linear combination

$$u = \sum_{k=1}^{\infty} c_k u_k, \quad \Rightarrow \quad u = c_1u_1 + c_2u_2 + \cdots + c_ku_k,$$

where the c_i, $i = 1, 2, \ldots, k$, are constants, is also a solution.

Superposition principle is the condition satisfied by a linear system.

One-dimensional wave equation



Consider a perfectly flexible elastic string stretched between two points at $x=0$ and $x=l$ with uniform tension T. If the string is displaced slightly from its initial position of rest and released, with the end points remaining fixed, then the string will vibrate. The position of any point P in the string will then depend on its distance from one end and on the instant in time. Its displacement u at any time t can thus be expressed as $u=f(x,t)$ where x is its distance from the left-hand end.

One-dimensional wave equation

The equation of motion is given by

$$\frac{d^2u}{dx^2} = \frac{1}{c^2} . \frac{d^2u}{dt^2} \qquad \text{where : } c^2 = \frac{T}{\rho}$$

in which T is the tension in the string and ρ the mass per unit length of the string. The displacement of the string is regarded as small so that T and ρ remain constant.

Now let us deal with the solution of this equation.

One-dimensional wave equation

The new equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2}$ has a solution $u(x, t)$.

Boundary conditions

(a) The string is fixed at both ends, i.e. at $x = 0$ and at $x = l$ for all values of time t. Therefore $u\,(x, t)$ becomes

$$\left.\begin{aligned} u(0, t) &= 0 \\ u(l, t) &= 0 \end{aligned}\right\} \quad \text{for all values of } t \geq 0$$

Initial conditions

(b) If the initial deflection of P at $t = 0$ is denoted by $f(x)$, then

$$u(x, 0) = f(x)$$

(c) Let the initial velocity of P be $g(x)$, then

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x)$$

So now we have listed all the information available from the question. Next we turn to solving the equation.

Solution of the wave equation

Solution by separation of variable.

Thus, we assume the solution is: *u(x,t)=X(x)T(t)* or be simplified as *u=XT*.

$$u = XT \quad \therefore \quad \frac{\partial u}{\partial x} = X'T \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T$$

$$\frac{\partial u}{\partial t} = XT' \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = XT''$$

The wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2}$ can then be written as

$$\boxed{X''T = \frac{1}{c^2} XT''}$$

and this can be transposed into $\frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T''}{T}$

Solution of the wave equation

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k$$

Notice that the left-hand side expression involves functions of x only and that the right-hand side expression involves functions of t only. Therefore, if these two expressions are to be equal for all values of the separate variables, then both expressions must be equal to a constant.

Denote this arbitrary constant by k. Then we have:

$$\frac{X''}{X} = k \quad \text{and} \quad \frac{1}{c^2} \cdot \frac{T''}{T} = k$$

$$\therefore \; X'' - kX = 0 \quad \text{and} \quad T'' - c^2kT = 0$$

Solution of the wave equation

Let us consider the first of these two equations for different values of k.

(1) *If* $k = 0, \quad X'' = 0 \quad \therefore\ X' = a \quad \therefore\ X = ax + b.$

$$\left.\begin{array}{l} \text{But } X = 0 \text{ at } x = 0 \quad \therefore\ b = 0 \quad \therefore\ X = ax \\ \text{and } X = 0 \text{ at } x = l \quad \therefore\ a = 0 \end{array}\right\} \therefore\ a = b = 0$$

$\therefore\ X = 0$ which is not oscillatory as the problem requires it to be.

(2) *If* k *is positive,* let $k = p^2 \quad \therefore\ X'' - p^2X = 0.$

The auxiliary equation is therefore $m^2 - p^2 = 0 \quad \therefore\ m^2 = p^2$

$$m = \pm p$$

$$\therefore\ X = Ae^{px} + Be^{-px}$$

But $X = 0$ at $x = 0 \quad \therefore\ 0 = A + B \quad \therefore\ B = -A$

and $X = 0$ at $x = l \quad \therefore\ 0 = Ae^{pl} - Ae^{-pl} \quad \therefore\ 0 = A(e^{pl} - e^{-pl})$

$$\therefore\ A = 0 \quad \therefore\ A = B = 0$$

Here again $X = 0$ which is not oscillatory.

(3) *If* k *is negative,* let $k = -p^2 \quad \therefore\ X'' + p^2X = 0.$

The auxiliary/characteristic equation has complex conjugate poles at $\pm jp$. The solution has the form:

$$X = A\cos px + B\sin px \qquad \longleftarrow \qquad (1)$$

which fits the requirements.

Solution of the wave equation

The second equation: $T'' - c^2 k T = 0$

Becomes…. $T'' + c^2 p^2 T = 0$

And the solution is: $T = C\cos(cpt) + D\sin(cpt)$ (2)

So, the overall solution *U*=*XT* is:

$$u(x,t) = (A\cos px + B\sin px)\,(C\cos cpt + D\sin cpt)$$

and, if we put $cp = \lambda$ $\quad \therefore\ p = \frac{\lambda}{c}$, this becomes

$$u(x,t) = \left(A\cos\frac{\lambda}{c}x + B\sin\frac{\lambda}{c}x\right)(C\cos\lambda t + D\sin\lambda t) \tag{3}$$

where *A*, *B*, *C*, *D* are arbitrary constants.

Solution of the wave equation

The result, of course, must satisfy the set of boundary conditions which we now turn to.

(a) $u = 0$ when $x = 0$ for all values of t. From this, we get: $A=0$

Because, substituting $u = 0$ and $x = 0$ in result (3) above

$$0 = A(C\cos\lambda t + D\sin\lambda t) \quad \text{for all } t \qquad \therefore \quad A = 0$$

$$\therefore \quad u(x,t) = B\sin\frac{\lambda}{c}x(C\cos\lambda t + D\sin\lambda t)$$

(b) $u = 0$ when $x = l$ for all t $\quad \therefore \quad 0 = B \sin\dfrac{\lambda l}{c}(C\cos\lambda t + D\sin\lambda t)$

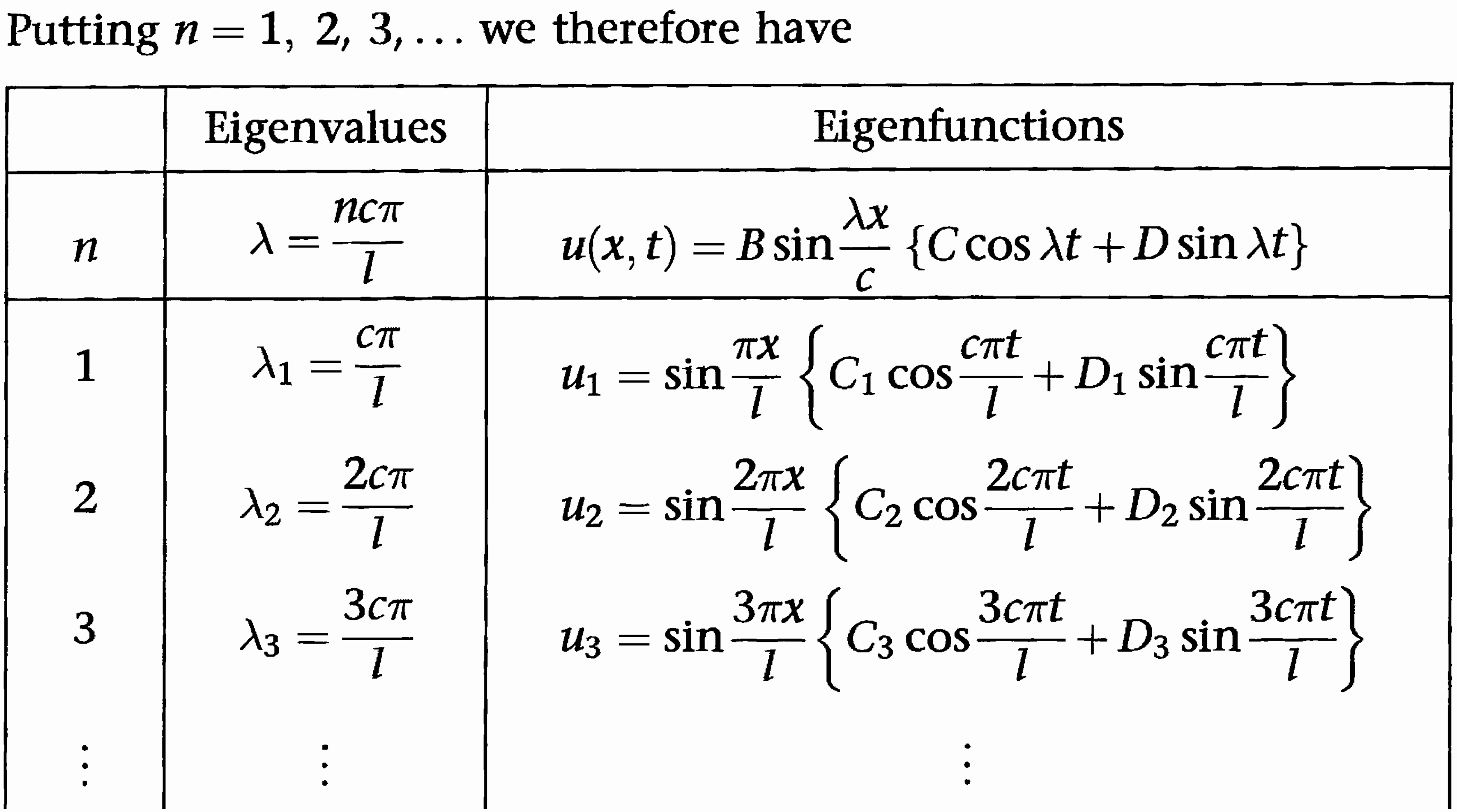
Now $B \neq 0$ or $u(x,t)$ would be identically zero. $\therefore \quad \sin\dfrac{\lambda l}{c} = 0.$

$$\therefore \quad \frac{\lambda l}{c} = n\pi \text{ where } n = 1,\ 2,\ 3, \ldots \qquad \therefore \quad \lambda = \frac{nc\pi}{l} \text{ for } n = 1,\ 2,\ 3, \ldots$$

Note that we exclude $n = 0$ since this would also make $u(x,\ t)$ identically zero.

Solution of the wave equation

As we can see, there is an infinite set of values of λ and each separate value gives a particular solution for *u(x,t)*. The values of λ are called the *eigenvalues* and each corresponding solution the *eigenfunction*.



Putting $n = 1,\ 2,\ 3, \ldots$ we therefore have

	Eigenvalues	Eigenfunctions
n	$\lambda = \frac{nc\pi}{l}$	$u(x,t) = B\sin\frac{\lambda x}{c}\{C\cos\lambda t + D\sin\lambda t\}$
1	$\lambda_1 = \frac{c\pi}{l}$	$u_1 = \sin\frac{\pi x}{l}\left\{C_1\cos\frac{c\pi t}{l} + D_1\sin\frac{c\pi t}{l}\right\}$
2	$\lambda_2 = \frac{2c\pi}{l}$	$u_2 = \sin\frac{2\pi x}{l}\left\{C_2\cos\frac{2c\pi t}{l} + D_2\sin\frac{2c\pi t}{l}\right\}$
3	$\lambda_3 = \frac{3c\pi}{l}$	$u_3 = \sin\frac{3\pi x}{l}\left\{C_3\cos\frac{3c\pi t}{l} + D_3\sin\frac{3c\pi t}{l}\right\}$
⋮	⋮	⋮

Solution of the wave equation

$$u_n = \sin\frac{n\pi x}{l}\left\{C_n \cos\frac{nc\pi t}{l} + D_n \sin\frac{nc\pi t}{l}\right\}$$

Note that the constant B has been absorbed into the constants C and D so that $BC = C_n$ and $BD = D_n$, where $C_1,\ C_2,\ C_3, \ldots$ and $D_1,\ D_2,\ D_3, \ldots$ are arbitrary constants.

Since the original wave equation is linear, we have already noted that if $u=u_1$, $u=u_2$, $u=u_3$,... are particular solutions, then from the superposition principle, a more general solution is:

$$\boxed{u = u_1 + u_2 + u_3 + \ldots.}$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{l}\left(C_n \cos\frac{nc\pi t}{l} + D_n \sin\frac{nc\pi t}{l}\right)\right\} \qquad (4)$$

Solution of the wave equation

We still have to find C_n and D_n and for this we use the initial conditions which we have not yet taken into account.

(c) At $t = 0$, $u(x, 0) = f(x)$ for $0 \leq x \leq l$

Therefore from (4), $u(x, 0) = f(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l}$

(d) Also at $t = 0$, $\left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x)$ for $0 \leq x \leq l$

We therefore differentiate (4) with respect to t and put $t = 0$, which gives

$$g(x) = \frac{c\pi}{l} \sum_{n=1}^{\infty} D_n n \sin \frac{n\pi x}{l}$$

Because…

Solution of the wave equation

$$\frac{du}{dt}=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\left\{-C_n\frac{nc\pi}{l}\sin\frac{nc\pi t}{l}+D_n\frac{nc\pi}{l}\cos\frac{nc\pi t}{l}\right\}$$

So, with $t=0$

$$\frac{du}{dt}=g(x)=\sum_{n=1}^{\infty}D_n\frac{nc\pi}{l}\sin\frac{n\pi x}{l}$$

Finally we need to involve Fourier series techniques to determine C_n and D_n so that...

$$C_n=\frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi}{l}x)\,dx$$

$$D_n=\frac{2}{nc\pi}\int_0^l g(x)\sin(\frac{n\pi}{l}x)\,dx$$

Solution of one-dimensional wave equation

SUMMARY

$$\frac{d^2u}{dx^2} = \frac{1}{c^2} \cdot \frac{d^2u}{dt^2}$$

The solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \sin \frac{n\pi x}{L} \left(B_n \cos \frac{nc\pi t}{L} + B_n^* \sin \frac{nc\pi t}{L} \right) \right\}$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L})\, dx$$

$$B_n^* = \frac{2}{nc\pi} \int_0^L g(x) \sin(\frac{n\pi x}{L})\, dx$$

That satisfies the initial and boundary conditions:

$$u(0,t) = 0$$

$$u(L,t) = 0$$

$u(x,0) = f(x)$ initial deflection

$\left.\frac{du}{dt}\right|_{t=0} = g(x)$ initial velocity

Examples 1-2

1. Find the solution of the wave equation corresponding to the triangular initial deflection *f(x)*:

$$f(x)=\begin{cases}\dfrac{2k}{L}x & 0<x<\dfrac{L}{2}\\ \dfrac{2k}{L}(L-x) & \dfrac{L}{2}<x<L\end{cases}$$

And initial velocity $g(x)=0$.

2. Find the solution $u(x, t)$ of the wave equation of the string with length $L=\pi$, $c=1$, initial velocity is 0, when the initial deflection is:

$$\left.\begin{array}{ll}\text{a.} & 0.001\sin(3x)\\ \text{b.} & k[\sin(x)-0.5\sin(2x)]\end{array}\right\}\ \text{Sinusoidal initial deflection}$$

Exercise

Repeat problem in number 2 with $L=1$ and $c=1$, and the initial deflection is $f(x)=kx(1-x)$.

Answer: $$u(x,t)=\frac{8k}{\pi^3}(\cos\pi t\sin\pi x+\frac{1}{27}\cos 3\pi t\sin 3\pi x+\frac{1}{125}\cos 5\pi t\sin 5\pi x+\ldots)$$